Borders bases in the rational Weyl algebra - with an application to the sunrise integral

Based on [2510.23411], joint with Anna-Laura Sattelberger

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Part I

Commutative Algebra

Refresher on commutative algebra 1/3

Here, multiplication is commutative

- A ring R is a set of elements closed under two binary operations: multiplication and addition.
  - Example: $P = \mathbb{C}[x, y]$, the polynomial ring in x and y.
- An ideal $I \subset R$ is a set of elements closed under addition, and under multiplication by elements $r \in R$ in the ring. Ideals can be *generated* by certain ring elements.
  - Example: In the ring of integers $\mathbb{Z}$, the ideal $I_2 = \langle 2 \rangle$ is the ideal generated by 2, and has all integer multiples of 2. I_2 contains the even integers.

Refresher on commutative algebra 2/3

- We can obtain quotient rings R/I by working with *equivalence classes* with the following equivalence relation: $a \sim b \iff a - b \in I$. These elements are called residue classes, and are denoted by $[a]$.
    - Intuition: In the quotient ring R/I, everything inside I is zero.
    - Example: $\mathbb{C}[x, y]/\langle 1 - xy \rangle = \mathbb{C}[x, 1/x]$.
- Working over a polynomial ring over a field k (e.g. $k = \mathbb{R}, \mathbb{C}$), the quotient rings are always vector spaces over k. If this vector space is finite-dimensional, we say that I is a zero-dimensional ideal.
    - Example: $\dim_{\mathbb{R}} \left(\mathbb{R}[x]/\langle 1 + x^2 \rangle\right) = 2$.

Refresher on commutative algebra 3/3

- When $I \subset R$ is zero-dimensional, we can give a basis to R/I . In such a basis, multiplication by the ring elements can be encoded by matrices, called *multiplication matrices*.

  - Example: We can choose a basis for $\mathbb{R}[x]/\langle 1 + x^2 \rangle$ with $\mathcal{O} = \{[1], [x]\}$.

    In this basis, multiplication by x is encoded in $M_x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

    We have $M_x \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $M_x \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

  - Remark: For polynomial rings over more variables, these multiplication matrices *commute*. After all, the ring is commutative.

A motivating example

For $R = \mathbb{C}[x, y]$

- Let $I = \langle x^2 + 2 + 2x - 2y, xy + 2x + y, y^2 + 2x + 2y \rangle$ be an ideal $I \subset R$.
- This ideal is zero-dimensional. A basis of R/I is given by $\mathcal{O} = \{1, x, y\}$.
- We can read off the multiplication matrices M_x, M_y in this basis:

$$M_x = \begin{pmatrix} 0 & -2 & 0 \\ 1 & -2 & -2 \\ 0 & 2 & -1 \end{pmatrix} \qquad M_y = \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & -2 \\ -1 & 1 & -2 \end{pmatrix}$$

- These matrices commute!

Another motivating example

For $R = \mathbb{C}[a, b, c, d, e, f, s, t, u][x, y]$

- Consider $I = \langle x^2 - a - bx - cy, xy - d - ex - fy, y^2 - s - tx - uy \rangle$
- Let's pretend that a basis of R/I is given by $\mathcal{O} = \{1, x, y\}$.
- We can read off the *formal* multiplication matrices M_x, M_y :

$$M_x = \begin{pmatrix} 0 & a & d \\ 1 & b & e \\ 0 & c & f \end{pmatrix} \qquad M_y = \begin{pmatrix} d & 0 & s \\ e & 0 & t \\ f & 1 & u \end{pmatrix}$$

- **Theorem** [See e.g. 4.3.17 in Kehrein, Kreuzer, Robbiano]: R/I is a vector space with basis given by $\mathcal{O}$ if, and only if, the matrices M_x and M_y commute.
  - This theorem arises in the theory of border bases.

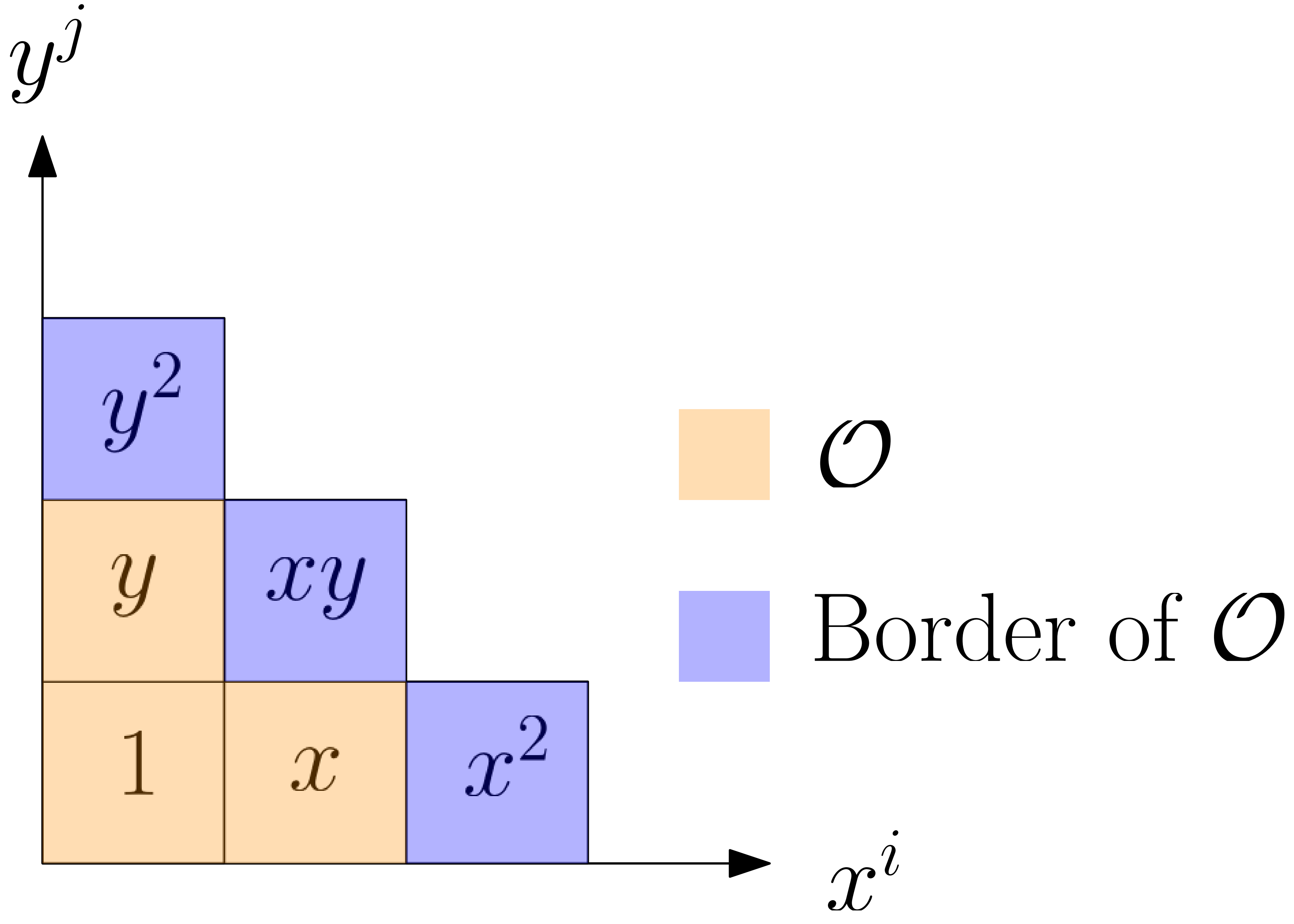
A first example of border bases

For $R = \mathbb{C}[a,b,c,d,e,f,s,t,u][x,y]$, $\mathcal{O} = \{1, x, y\}$

- We say the ideal $I = \langle \underline{x^2} - a - bx - cy, \underline{xy} - d - ex - fy, \underline{y^2} - s - tx - uy \rangle$ is generated by an $\mathcal{O}$-border basis, when the residue classes of $\mathcal{O} = \{1, x, y\}$ form a basis for R/I. Here, the border of $\mathcal{O}$ is the set $\partial\mathcal{O} = \{x^2, xy, y^2\}$.

- Why these names? Look at the diagram!

$$M_x = \begin{pmatrix} 0 & a & d \\ 1 & b & e \\ 0 & c & f \end{pmatrix} \quad M_y = \begin{pmatrix} d & 0 & s \\ e & 0 & t \\ f & 1 & u \end{pmatrix}$$



Part II

Weyl algebras and border bases

The Weyl algebra

A polynomial ring of differential operators

- The n-th Weyl algebra is given by $D_n = \mathbb{C}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle$, and all the variables are pairwise commuting except for: $[\partial_i , x_j] = \delta_{ij}$.
- Notation:
  - We will always write operators with the derivatives at the very right, i.e. for $P = \partial_x x - 2 = x\partial_x - 1$, we prefer the second form of this operator.
  - We use $\bullet$ to denote the application of a differential operator on a function, e.g. $\partial_x \bullet f = \partial f / \partial x$.

The rational Weyl algebra

- The n-th rational Weyl algebra is given by $R_n = \mathbb{C}(x_1, \ldots, x_n)\langle \partial_1, \ldots, \partial_n \rangle$. It consists of differential operators with coefficients in the field of rational functions.
- Both D_n and R_n are non-commutative rings. Due to this, we will consider *left*-ideals $I \subset D_n$ (resp. $J \subset R_n$). These are sets of differential operators closed under addition and *left* multiplication by any elements in D_n (resp. R_n).
- These ideals encode linear PDEs!
  - For example, $J = \langle x\partial - 2 \rangle$ contains all the differential operators that annihilate $f(x) = x^2$. I.e. $[p(x\partial - 2)] \bullet f = 0, \ \forall p \in R_1$.
  - In the case of Feynman integrals $\mathcal{I}$, the ideal $\mathrm{Ann}(\mathcal{I})$ of all differential operators that annihilate $\mathcal{I}$ is of particular interest.

Quotients of left-ideals

For $J \subset R_n$ left-ideals

- R_n/J consists of equivalence classes, where $a \sim b \iff a - b \in J$, and has a natural left-action by R_n. I.e., they are R_n-modules.

- R_n/J has the structure of a $\mathbb{C}(x_1, \ldots, x_n)$-vector space. We write $\operatorname{rank}(J) = \dim_{\mathbb{C}(x_1,\ldots,x_n)} R_n/J = m$. If $m \in \mathbb{N}$, we say that J has holonomic rank m.
  - Example: $J = \langle \partial_x^2 + 1, \partial_y^2 + 1 \rangle \in R_2$ has holonomic rank 4. A convenient basis for R_2/J is given by $\mathcal{O} = \{1, \partial_x, \partial_y, \partial_x \partial_y\}$.
  - Intuition: The holonomic rank m will tell us the number of solutions to the linear PDE encoded by J.

Order ideals $\mathcal{O}$

Encoding bases of R_n/J

- Let $\mathbb{T}^n$ be the set of monomials in partial derivatives $\partial_1, \ldots, \partial_n$, including 1, endowed with multiplication (e.g. $\partial_1\partial_2^2$ and ∂_1^7 are monomials).
- Definition: An order ideal $\mathcal{O}$ is a subset of $\mathbb{T}^n$ that contains 1, and that is closed under taking divisors. I.e., if $a \in \mathcal{O}$ and $b \,|\, a$, then $b \in \mathcal{O}$. We will be interested in finite order ideals, $|\mathcal{O}| = m$.
  - Example: $\mathcal{O} = \{1, \partial_x, \partial_y, \partial_x\partial_y\}$ is an order ideal. Note that because $\partial_x\partial_y \in \mathcal{O}$, the other 3 elements also have to be there.
- If$\mathcal{O}$ is a basis of R_n/J, we can find matrices encoding the left-action of ∂_i.

Method 1: Connection matrices

For $J \subset R_n$ with $\operatorname{rank}(J) = m$ and $\mathscr{O}$ a $\mathbb{C}(x_1, \ldots, x_n)$-basis of R_n/J

- Let $\mathscr{O} = \{t_1, t_2, \ldots, t_m\}$. Without loss of generality, we can choose $t_1 = 1$.
- Let $f \in \operatorname{Sol}(J)$ be a generic solution of J. We can form a vector $F = (f, t_2 \bullet f, \ldots, t_m \bullet f)^\top$. This vector satisfies a 1st order linear PDE, called Pfaffian system:

$$\partial_i \bullet F = A_i F \,, \qquad \text{for } i = 1, \ldots, n.$$

- The matrices $A_i \in \operatorname{Mat}_{m\times m}\big(\mathbb{C}(x_1, \ldots, x_n)\big)$ are called *connection matrices*. They satisfy the integrability conditions:

$$[A_i, A_j] = \partial_i \bullet A_j - \partial_j \bullet A_i \,, \qquad \text{for } i, j = 1, \ldots, n.$$

Gauge transformations and connection matrices

For $J \subset R_n$ with $\operatorname{rank}(J) = m$ and $\mathcal{O}$ a $\mathbb{C}(x_1, \ldots, x_n)$-basis of R_n/J

- If $\widetilde{F} = gF$, for $g \in \mathrm{GL}_m\left(\mathbb{C}(x_1, \ldots, x_n)\right)$, then $\widetilde{F}$ satisfies a Pfaffian system:

$$\partial_i \bullet \widetilde{F} = \widetilde{A}_i \widetilde{F}\ , \qquad \text{for } i = 1,\ldots, n,$$

- The gauge-transformed matrices $\widetilde{A}_i$ are given by:

$$\widetilde{A}_i = g A_i g^{-1} + \frac{\partial g}{\partial x_i} g^{-1}\ , \quad \text{for} \quad i = 1,\ldots, n\,.$$

- Remark: The above transformation ensures that the $\widetilde{A}_i$ satisfy the integrability conditions.

Method 2: Multiplication matrices M_{∂_i}

For $J \subset R_n$ with $\operatorname{rank}(J) = m$, and $\mathcal{O} = \{t_1, \ldots, t_m\}$ a $\mathbb{C}(x_1, \ldots, x_n)$-basis of R_n/J

- We can encode the of left-multiplication by ∂_i on R_n/J by matrices $M_{\partial_i} \in \mathrm{Mat}_{m\times m}\left(\mathbb{C}(x_1, \ldots, x_n)\right)$.
- To obtain M_{∂_i}, note that we can write $\partial_i t_k = \sum_{i=1}^{l} c_{kl}^{(i)}[t_l]$, with $c_{kl}^{(i)} \in \mathbb{C}(x_1, \ldots, x_n)$. The k-th column of M_{∂_i} is given by $\left(c_{k1}^{(i)}, c_{k2}^{(i)}, \ldots, c_{km}^{(i)}\right)^\top$.
- These matrices satisfy the <u>integrability conditions</u>:

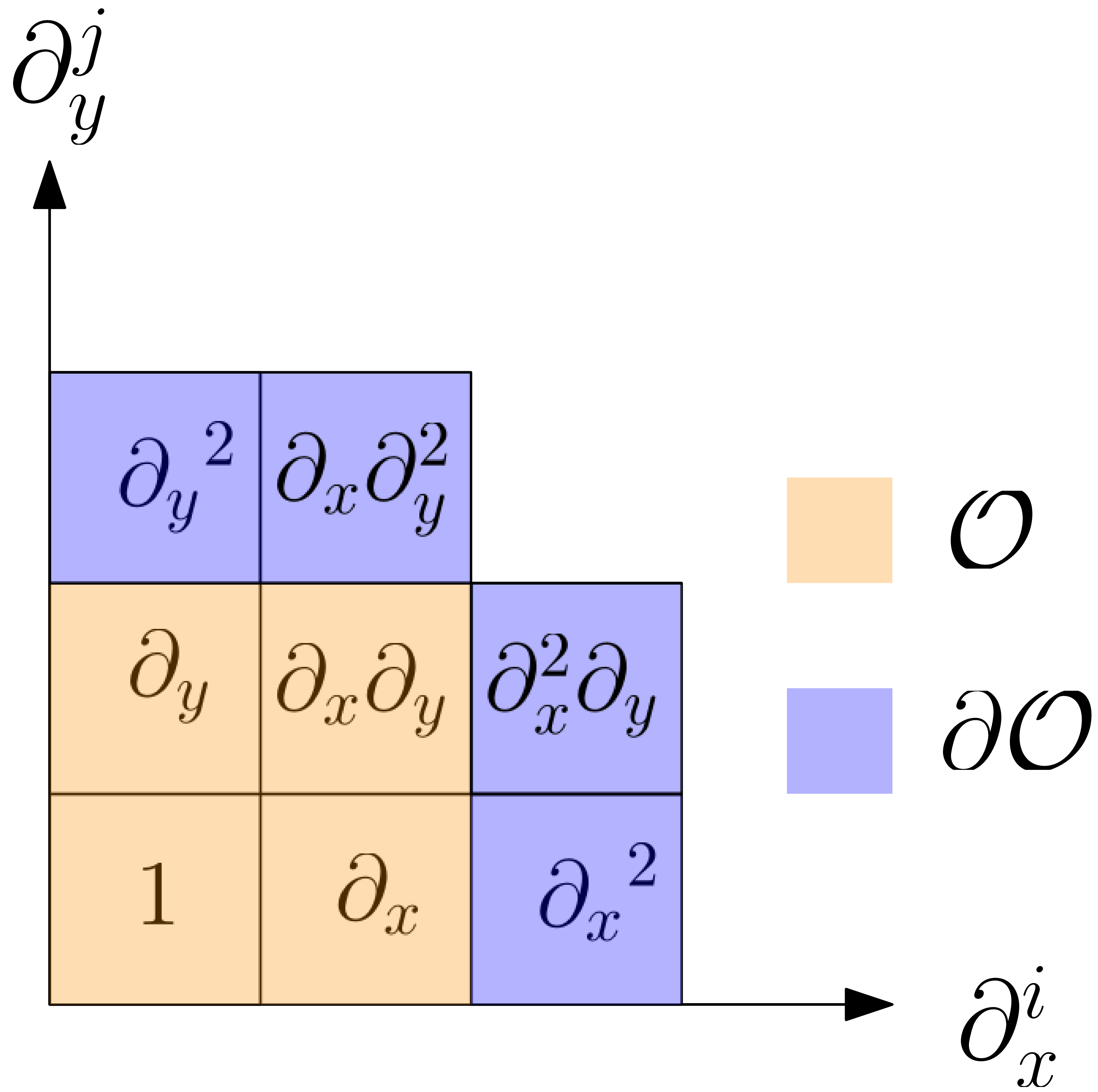
$$[M_{\partial_i}, M_{\partial_j}] = \partial_j \bullet M_i - \partial_i \bullet M_j\,, \qquad \text{for } \; i, j = 1,\ldots, n.$$

- Multiplication matrices related to connection matrices by a transpose:

$$M_{\partial_i}^\top = A_i \;.$$

Order ideals and their borders

- Definition: The border of $\mathcal{O}$ is denoted by $\partial\mathcal{O}$, and is given by: $\partial\mathcal{O} = \left(\cup_{i=1}^{n} \partial_i \mathcal{O} \right) \setminus \mathcal{O}$.
  - Example: For $\mathcal{O} = \{1, \partial_x, \partial_y, \partial_x\partial_y\}$, the border is given by $\partial\mathcal{O} = \{\partial_x^2, \partial_x^2\partial_y, \partial_x\partial_y^2, \partial_y^2\}$.
- Intuition: In the 2-variable case, if an order ideal contains a monomial $\partial_x^n\partial_y^m$, it also contains all monomials to the left of and below $\partial_x^n\partial_y^m$.



Border prebases in the rational Weyl algebra

For $\mathcal{O}$ an order ideal

- Let $\mathcal{O} = \{t_1, t_2, \dots, t_m\}$ and $\partial\mathcal{O} = \{b_1, b_2, \dots, b_p\}$.
- An <u>$\mathcal{O}$-border prebasis</u> $G \subset R_n$ is of the form $G = \{g_1, \dots, g_p\}$, where

  $$g_i = \underline{b_i} - \sum_{j=1}^{m} c_{i,j}\, t_j \ ,$$

  where $c_{i,j} \in \mathbb{C}(x_1, \dots, x_n)$.
- We can read off *formal* multiplication matrices M_{∂_i} , $i = 1,\dots,n$. These encode left-multiplication by ∂_i on R_n/J, where $J = \langle G \rangle$, under the assumption that the residue classes of $\mathcal{O}$ form a basis of R_n/J. The entries in M_{∂_i} are either 0, 1 or some $c_{j,k}$.

Border bases in the rational Weyl algebra

For $\mathcal{O}$ an order ideal with $|\mathcal{O}| = m$, $J \subset R_n$

- Let $G \subset R_n$ be an $\mathcal{O}$-border prebasis.
- Definition: an $\mathcal{O}$-border prebasis $G \subset J$ is an $\mathcal{O}$-border basis if the residue classes of $\mathcal{O}$ form a $\mathbb{C}(x_1, \ldots, x_n)$-basis of R_n/J.
  - Remark. If $G \subset J$ is an $\mathcal{O}$-border basis, it generates J. We write $J_{\mathcal{O}} = \langle G \rangle$.
- Theorem (R, Sattelberger '25): An $\mathcal{O}$-border prebasis G is an $\mathcal{O}$-border basis if, and only if, the formal multiplication matrices M_{∂_i} satisfy the integrability conditions.

Part 3: The Sunrise integral

The Sunrise integral family

In dimension d

- The unequal-mass Sunrise integrals belongs to the Feynman integral family $\mathcal{I}_{\nu_1,\nu_2,\nu_3,\nu_4,\nu_5}$ given by:

$$\mathcal{I}_{\nu_1,\nu_2,\nu_3,\nu_4,\nu_5} = \int \prod_{j=1}^{2} \frac{\mathrm{d}^d k_j}{i\pi^{d/2}} \frac{(k_1 \cdot p)^{-\nu_4}(k_2 \cdot p)^{-\nu_5}}{(k_1^2 - m_1^2)^{\nu_1}(k_2^2 - m_2^2)^{\nu_2}(p - k_1 - k_2)^2 - m_3^2)^{\nu_3}} ,$$

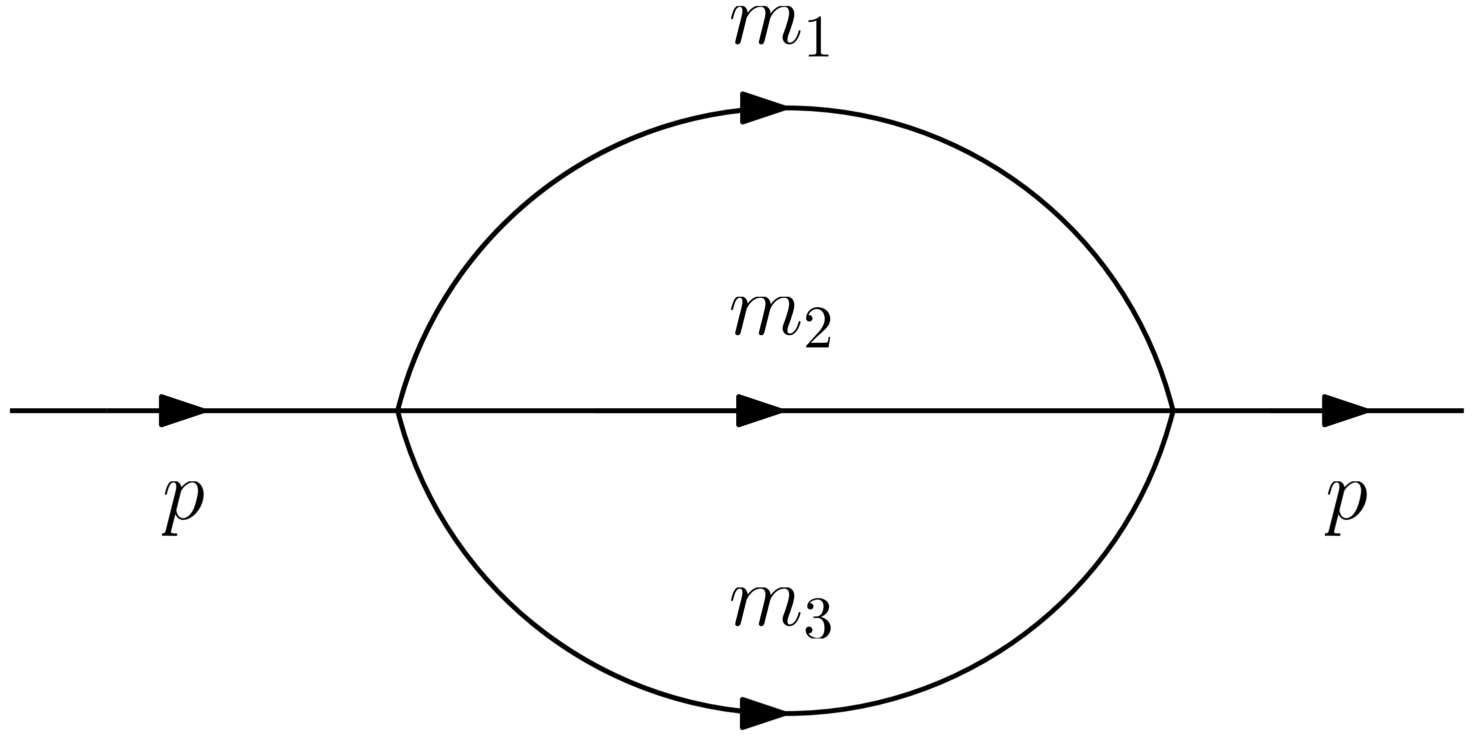
  where $\nu_i \in \mathbb{Z}$, $m_i \in \mathbb{C}$, $p \in \mathbb{R}^{1,3}$, and we integrate over the loop momenta $k_j \in \mathbb{R}^{1,3}$.

- We have $\mathcal{I}_{\nu_1,\ldots,\nu_5} = \mathcal{I}_{\nu_1,\ldots,\nu_5}(d; s, m_1^2, m_2^2, m_3^2)$.

The Sunrise integral

In dimension d

- The sunrise integral $\mathscr{I}_{1,1,1,0,0}$ is the integral corresponding to the Feynman diagram:



- For convenience, we will use the variables $\{x_0, x_1, x_2, x_3\} = \{s, m_1^2, m_2^2, m_3^2\}$.
- We will work over $R_4(d) = \mathbb{C}(d)[x_0, x_1, x_2, x_3]\langle \partial_0, \partial_1, \partial_2, \partial_3 \rangle$.

A differential equation for the Sunrise integral

- Following [Maggio, Sohnle, 2504.17757], the sunrise integral sits in a vector of 7 masters integrals:

$$F = (\mathcal{I}_{1,1,0,0,0}, \mathcal{I}_{1,0,1,0,0}, \mathcal{I}_{0,1,1,0,0}, \mathcal{I}_{1,1,1,0,0}, \mathcal{I}_{2,1,1,0,0}, \mathcal{I}_{1,2,1,0,0} \cdot \mathcal{I}_{1,1,2,0,0})^{\top}.$$

$$= (\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4, \mathcal{I}_5, \mathcal{I}_6, \mathcal{I}_7)^{\top}.$$

- The Pfaffian system is of the form

$$\partial_i \bullet F = A_i \cdot F \,, \qquad \text{for } i = 0,1,2,3 \,,$$

where $A_i \in \mathrm{Mat}_{7\times 7}\big(\mathbb{C}(d, x_0, x_1, x_2, x_3)\big)$.

A change of basis for F

- We now consider the following change of basis for F:

$$\widetilde{F} = \begin{pmatrix} \mathscr{I}_4 \\ \partial_1 \bullet \mathscr{I}_4 \\ \partial_2 \bullet \mathscr{I}_4 \\ \partial_3 \bullet \mathscr{I}_4 \\ \partial_1\partial_3 \bullet \mathscr{I}_4 \\ \partial_2\partial_3 \bullet \mathscr{I}_4 \\ \partial_3^2 \bullet \mathscr{I}_4 \end{pmatrix} = g_{\text{sunrise}} \begin{pmatrix} \mathscr{I}_1 \\ \mathscr{I}_2 \\ \mathscr{I}_3 \\ \mathscr{I}_4 \\ \mathscr{I}_5 \\ \mathscr{I}_6 \\ \mathscr{I}_7 \end{pmatrix},$$

  where $g_{\text{sunrise}} \in \mathrm{GL}_7\left(\mathbb{C}(d)(x_0, x_1, x_2, x_3)\right)$.

- Note that $\mathscr{O} = \{1, \partial_1, \partial_2, \partial_3, \partial_1\partial_3, \partial_2\partial_3, \partial_3^2\}$ here is an order ideal!

A gauge transform of the sunrise Pfaffian system

- The new vector $\widetilde{F}$ satisfies a Pfaffian system:

$$\partial_i \bullet \widetilde{F} = \widetilde{A}_i \widetilde{F} \,, \qquad\qquad \text{for } i = 0,1,2,3 \,.$$

- Proposition: From the Pfaffian system above, we can read off an $\mathcal{O}$-border prebasis G of differential operators that annihilate the sunrise integral. Because this is an integrable connection, this is actually an $\mathcal{O}$-border basis. We denote $J_{\mathcal{O}} = \langle G \rangle$.

- We have that $\operatorname{rank}(J_{\mathcal{O}}) = 7$.

- **That's it!**

Some more details on $J_{\mathcal{O}} = \langle G \rangle$

- $\mathcal{O} = \{1, \partial_1, \partial_2, \partial_3, \partial_1\partial_3, \partial_2\partial_3, \partial_3^2\}$, and $\partial\mathcal{O} = \{\partial_0, \partial_0\partial_1, \ldots, \partial_3^3\}$, with $|\partial\mathcal{O}| = 16$.
- Example: The 1st row of $\widetilde{A}_0$ is

$$\left(\frac{d-3}{x_0}, -\frac{x_1}{x_0}, -\frac{x_2}{x_0}, -\frac{x_3}{x_0}, 0, 0, 0 \right) ,$$

  from which we read off

$$b_1 = \underline{\partial_0} - \frac{d-3}{x_0} + \frac{x_1}{x_0}\partial_1 + \frac{x_2}{x_0}\partial_2 + \frac{x_3}{x_0}\partial_3 \, .$$

- The ideal $J_{\mathcal{O}}$ coincides with the one described in a recent work of Flieger [Fleiger, 2508.04309] for banana integrals.

Concluding remarks

- We have Mathematica notebooks available describing all the computations presented here!
- Border bases in the rational Weyl algebra allow us to go from Pfaffian systems (differential equations) to R_n-ideals. This complements tools such as Macaulay Matrices [Chestnov et al., 2204.12983] or the ConnectionMatrices package in Macaulay2 [Görlach et al, 2504.01362], which compute Pfaffian systems from D_n -ideals.
- Border bases give new tools to characterize ideals $J \in R_n$ of finite holonomic rank.
- Moreover, we can use tools such as LiteRed [Lee, Roman N.,1310.1145], which compute differential equations of Feynman integrals to study their annihilating ideals.

Annotated Bibliography

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Thanks for your attention!

Feel free to ask more questions

- I'm on the postdoc market this year. If you're interested, let me know :)

  See more in: pilk.love/carlos